Open manifold with nonnegative Ricci curvature and collapsing volume

Jing Mao

Departamento de Matemática, Instituto Superior Técnico, Technical University of Lisbon, Edifício Ciência, Piso 3, Av. Rovisco Pais, 1049-001 Lisboa, Portugal; jiner120@tom.com

Abstract

In this paper, an n-dimensional complete open manifold with nonnegative Ricci curvature and collapsing volume has been investigated. If its radial sectional curvature bounded from below, it shows that such a manifold is of finite topological type under some restrictions shown below.

1 Introduction

Without specification, in this paper we let (M,g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature, denote by $B(p,r)$ the open geodesic ball centered at a point $p\in M$ with radius r and $\mathrm{vol}(B(p,r))$ its volume, and let w_n be the volume of unit ball in the Euclidean space R^n. By the classical volume comparison theorem [6, 7], we know the function $r\to\frac{\mathrm{vol}(B(p,r))}{w_n r^n}$ is monotone decreasing. Define

$$\alpha_M := \lim_{r\to\infty}\frac{\mathrm{vol}(B(p,r))}{w_n r^n},$$

it is not difficult to prove α_M is independent of the choice of p, which implies α_M is a global geometric invariant. Obviously, $\alpha_M\in[0,1]$, and

$$\alpha_M w_n r^n \le \mathrm{vol}(B(p,r)) \le w_n r^n, \quad for\ \forall p\in M\ \ and\ \ \forall r>0. \tag{1.1}$$

We say that (M,g) has large volume growth provided $\alpha_M>0$. Riemannian manifold with nonnegative Ricci curvature and large volume growth has been investigated intensively and some good results have been obtained in the past decades. Let (N,g) be an n-dimensional complete open manifold with Ricci curvature $Ric_N\ge 0$ and $\alpha_N>0$. By Bishop-Gromov comparison theorem [6, 7], N^n is isometric to R^n when $\alpha_N=1$. It has been shown by Li [8] that N has finite fundamental group. Anderson [9] has proved that the order of $\pi_1(N)$ is bounded from above by $\frac{1}{\alpha_N}$. Petersen has

MSC 2010: 53C20; 53C21
Key Words: Critical point; Collapsing volume; Radial sectional curvature; Finite topological type.
supported by Fundação de Ciência e Tecnologia (FCT) through a doctoral fellowship SFRH/BD/60313/2009.

conjectured that if $\alpha_N > \frac{1}{2}$, then N^n is diffeomorphic to R^n in [10]. A theorem has been proved by do Carmo and Xia in [11] to support this conjecture. Xia [1, 2] has shown that N^n is diffeomorphic to a Euclidean space R^n or is of finite topological type under different restrictions about α_N and volume ratio $\frac{\mathrm{vol}(B(p,r))}{w_n r^n}$.

It seems that if we want to get conclusions like M^n is diffeomorphic to a Euclidean space R^n or is of finite topological type for an n-dimensional complete open Riemannian manifold (M,g) with nonnegative Ricci curvature, the large volume growth condition can not be avoided. However, in this paper, we find that we could use the α-order collapsing volume condition to replace the large volume growth condition. Actually, for M^n with nonnegative Ricci curvature, we have

$$c_1(n)vol(B(p,1))r \leq vol(B(p,r)) \leq w_n r^n,$$

where $c_1(n)$ is a constant depending on n. This inspires us that maybe we could consider some restriction on $\mathrm{vol}(B(p,1))$ if we want to get conclusions similar with those in [1], since $\mathrm{vol}(B(p,1))$ has connection with the volume ratio $\frac{\mathrm{vol}(B(p,r))}{w_n r^n}$. Fortunately, in this paper we prove that this is possible. Now, we want to give our main result in this paper, however, before that we need to introduce some concepts and conclusions. First, we need to use the following notion introduced in [3, 4]

Definition 1.1. *Let M be a complete noncompact manifold and let $p \in M$ be a point such that*

$$v_p(r) = \inf_{x \in S(p,r)} \mathrm{vol}(B(x,1)) = \mathrm{O}\left(\frac{1}{r^\alpha}\right), \tag{1.2}$$

where $S(p,r)$ denotes the geodesic sphere centered at p with radius r on M, then we say that M has α-order collapsing volume.

We also need the following lemma in [4]

Lemma 1.2. *Let M be a complete noncompact n-manifold with nonnegative Ricci curvature $Ric_M \geq 0$. Then there is a constant c_2 such that for $\forall R \geq r$, we have*

$$vol(B(p,R) \setminus B(p,r)) \leq c_2 \int_r^R \frac{\mathrm{vol}(B(p,s))}{s} ds. \tag{1.3}$$

A manifold M is said to have finite topological type if there is a compact domain Ω whose boundary $\partial\Omega$ is a topological manifold such that $M \setminus \Omega$ is homeomorphic to $\partial\Omega \times [0,\infty)$. For a fixed point $p \in M$, we say its radial sectional curvature, K_p^{min}, bounded from below by a constant $-C$ if for any minimal geodesic γ starting from p all sectional curvatures of the planes which are tangent to γ are greater than or equal to $-C$, i.e. $K_p^{min} \geq -C$. The main result is the following

Theorem 1.3. *Let (M,g) be an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature $Ric_M \geq 0$ and α-order collapsing volume ($0 \leq \alpha \leq \frac{1}{n}$). Suppose that $K_p^{min} \geq -C$ for some point $p \in M$ and some positive constant C. If*

$$\limsup_{r \to \infty} \left[\frac{\mathrm{vol}(B(p,r))}{r^{1+\frac{1}{n}-\alpha}}\right] \leq c_3 \left(\frac{\log 2}{8\sqrt{C}}\right)^{\frac{n-1}{n}}, \tag{1.4}$$

where c_3 is a positive constant depending on c_2, then M has finite topological type.

Our Theorem 1.3 is a generalization of Zhan's main theorem 3 in [3], since we only need the radial sectional curvature K_p^{min} bounded from below. Besides, we will give a more general version of our main theorem in the last section, which is a generalization of theorem 8 in [3] and shows the advantage of our result indeed. The paper is organized as follows. Some useful conclusions will be introduced and proved in Section 2. In Section 3, we will give the proof of Theorem 1.3.

2 Some useful facts

First, we would like to give an isotopy lemma obtained by Grove and Shiohama which will be used in the proof of our main theorem later.

Lemma 2.1. *([12]) If $r_1 \leq r_2 \leq \infty$ and a connected component D of $\overline{B(p,r_2)} \setminus B(p,r_1)$ is free of critical points of p, then D is homeomorphic to $D_1 \times [r_1,r_2]$, where D_1 is a topological submanifold without boundary.*

For convenience, throughout this paper, all geodesics are assumed to have unit speed. In order to have a topological cognizance of the above lemma, we want to recall the notion of critical point here. For a point $p \in M$, let $d_p(x) = d(p,x)$, where d is the metric on the Riemannian manifold M, obviously, the function d_p is Lipschitz continuous, however, it is not a smooth function on the cut locus of p, which implies the critical points of d_p can not be defined in a usual way. The notion of critical points of d_p was introduced minutely in [12]. A point $q \in M$ different from p is called a critical point of d_p if there always exists a minimizing geodesic γ from q to p such that for any $v \in T_qM$, the forming angle $\angle(v,\gamma'(0))$ satisfies $\angle(v,\gamma'(0)) \leq \frac{\pi}{2}$. We simply say q is a critical point of p. By the above isotopy lemma, we know that an n-dimensional complete noncompact Riemannian manifold M is diffeomorphic to a Euclidean space R^n if there is a point $p \in M$ such that p has no critical points other than p, which shows the importance of this lemma.

Now, we want to recall a notion named k-th Ricci curvature ($1 \leq k \leq n-1$) for the n-dimensional Riemannian manifold M^n. We say that the k-th Ricci curvature of M is nonnegative if for any point $x \in M$ and any mutually orthogonal unit tangent vector $e, e_1, \cdots, e_k \in T_xM$, we have $\sum_{i=1}^k K(e \wedge e_i) \geq 0$, here $K(e \wedge e_i)$ is the sectional curvature of the plane spanned by e and e_i. Denote this fact by $Ric_M^{(k)} \geq 0$. Notice that if $Ric_M^{(k)} \geq 0$, then $Ric_M \geq 0$. Let $p,q \in M$, then the excess function is defined by

$$e_{pq}(x) := d(p,x) + d(q,x) - d(p,q).$$

We have the following lemma which gives an upper bound for the excess function.

Lemma 2.2. *([13, 14]) Let (M,g) be an n-dimensional complete Riemannian manifold with $Ric_M^{(k)} \geq 0$ for some $1 \leq k \leq n-1$. Let $\gamma : [0,a] \to M$ be a minimal geodesic from p to q. Then for any $x \in M$,*

$$e_{pq}(x) \leq 8 \left(\frac{s^{k+1}}{r} \right)^{\frac{1}{k}}, \tag{2.1}$$

where $s = d(x,\gamma)$, $r = \min(d(p,x), d(q,x))$.

Let $\gamma : [0,\infty) \to M$ be a ray emanating from p. For any $x \in M$, by triangle inequality, it is easy to see that $e_{p,\gamma(t)}(x) = d(p,x) + d(\gamma(t),x) - t$ is decreasing in t and that $e_{p,\gamma(t)}(x) \geq 0$. Define the excess function $e_{p,\gamma}$ associated to p and γ as

$$e_{p,\gamma}(x) = \lim_{t\to\infty} e_{p,\gamma(t)}(x). \tag{2.2}$$

Obviously, $e_{p,\gamma}(x) \leq e_{p,\gamma(t)}(x)$ for any $t > 0$. For this function $e_{p,\gamma}$, Xia [1] has proved the following lemma.

Lemma 2.3. *Let (M,g) be a complete open Riemannian manifold with $K_p^{min} \geq -C$ for some $C > 0$ and $p \in M$. Suppose $x \neq p$ is a critical point of p. Then for any ray $\gamma : [0,\infty) \to M$ issuing from p, we have*

$$e_{p,\gamma}(x) \geq \frac{1}{\sqrt{C}} \log\left(\frac{2}{1+e^{-2\sqrt{C}d(p,x)}}\right). \tag{2.3}$$

At the end of this section, we want to give a lemma which will play an important role in the proof of our main theorem. In order to prove the lemma, we have to give some notions first. Let M be an n-dimensional complete open Riemannian manifold. For a given point $p \in M$, set

$$v_p(A,r) = \inf_{x\in S(p,r)} \mathrm{vol}\left(B\left(x,\frac{A}{2}\right)\right), \qquad 0 < A < \frac{r}{2}, \tag{2.4}$$

here $S(p,r)$ has the same meaning as before. For $r > 0$ and a point $p \in M$, let

$$R(p,r) = \{\gamma(r) | \gamma \textit{ is a ray from } p\},$$

obviously, $R(p,r)$ is the set of points of the intersections of the geodesic sphere centered at p of radius r with all the rays issuing from p. Let

$$R_p(x) = d(x,R(p,r)), \quad \text{where } \ r = d(p,x). \tag{2.5}$$

We can prove

Lemma 2.4. *Let M be an n-dimensional complete noncompact Riemannian manifold with non-negative Ricci curvature $Ric_M \geq 0$, and let $p \in M$, then for any $r > 4$ and $x \in S(p,r)$, we have*

$$d(x,R_p) \leq c_4 \frac{\mathrm{vol}(B(p,r))\cdot(r+2)^n}{v_p(r)\cdot r^n} \log\left(\frac{r+2}{r-2}\right), \tag{2.6}$$

where $c_4 = 8c_2$ is a positive constant depending only on c_2, $v_p(r) := v_p(2,r)$, and R_p denotes the union of rays issuing from p.

Proof. Here we use a similar method as that of lemma 6 in [3]. Let Ω_r be a boundary component of $M \setminus \overline{B(p,r)}$ with $\Omega_r \cap R(p,r) \neq \emptyset$. So, there exists a ray γ_p such that $\gamma_p(r) \in \Omega_r$. Let $\{B(p_j,\frac{A}{2})\}$ be a maximal set of disjoint balls with radius $\frac{A}{2}$ and center $p_j \in \Omega_r$, here $0 < A < \frac{r}{2}$, then

$$\bigcup_{j=1}^{N} B(p_j,A) \supset \Omega_r$$

and

$$N \leq \frac{\mathrm{vol}(B(p,r+A) \setminus B(p,r-A))}{v_p(A,r)},$$

where $v_p(A,r)$ is defined as (2.4). By the connectedness of Ω_r, we know that for any point $x \in \Omega_r$, there exists a subset of $\{p_j\}_{j=1,\cdots,N}$, say $\{q_1,\cdots,q_k\}$, $k \leq N$, such that

$$B(q_l,A) \cap B(q_{l+1},A) \neq \emptyset, \quad 1 \leq l \leq k-1,$$

and $x \in B(q_1,A)$, $\gamma_p(r) \in B(q_k,A)$. Hence, we can easily construct a piecewise smooth geodesic c joining x and $\gamma_p(r)$ through q_l's. This implies

$$d(x,\gamma_p(r)) \leq L(c) \leq 4NA \leq 4 \cdot \frac{\mathrm{vol}(B(p,r+A) \setminus B(p,r-A))}{v_p(A,r)} \cdot A, \tag{2.7}$$

where $L(c)$ denotes the length of c. Then, by (2.5), we have

$$R_p(x) \leq 4 \cdot \frac{\mathrm{vol}(B(p,r+A) \setminus B(p,r-A))}{v_p(A,r)} \cdot A,$$

moreover,

$$d(x,R_p) \leq R_p(x) \leq 4 \cdot \frac{\mathrm{vol}(B(p,r+A) \setminus B(p,r-A))}{v_p(A,r)} \cdot A, \tag{2.8}$$

since $R(p,x)$ is only a part of the point set R_p.

On the other hand, By Lemma 1.2, we can obtain

$$\begin{aligned} \frac{\mathrm{vol}(B(p,r+A) \setminus B(p,r-A))}{v_p(A,r)} &\leq \frac{c_2}{v_p(A,r)} \int_{r-A}^{r+A} \frac{\mathrm{vol}(B(p,s))}{s} ds \\ &\leq c_2 \cdot \frac{\mathrm{vol}(B(p,r+A))}{v_p(A,r)} \log\left(\frac{r+A}{r-A}\right), \end{aligned}$$

furthermore, together with (2.8), we have

$$d(x,R_p) \leq 4c_2 A \cdot \frac{\mathrm{vol}(B(p,r+A))}{v_p(A,r)} \log\left(\frac{r+A}{r-A}\right).$$

Choose $A = 2$, then we get

$$d(x,R_p) \leq c_4 \cdot \frac{\mathrm{vol}(B(p,r+2))}{v_p(r)} \log\left(\frac{r+2}{r-2}\right), \tag{2.9}$$

here $c_4 = 8c_2$. By the volume comparison theorem (see [6, 7]), the expression (2.9) becomes

$$d(x,R_p) \leq c_4 \cdot \frac{\mathrm{vol}(B(p,r+2))}{v_p(r)} \log\left(\frac{r+2}{r-2}\right) \leq c_4 \cdot \frac{\mathrm{vol}(B(p,r)) \cdot (r+2)^n}{v_p(r) \cdot r^n} \log\left(\frac{r+2}{r-2}\right),$$

which implies our lemma. □

Remark 2.5. Here we would like to point out that Lemma 2.4 is still true if we reduce the condition $Ric_M \geq 0$ to $Ric_M^{min} \geq 0$, the radial Ricci curvature is nonnegative. This is because for a complete open Riemannian manifold with nonnegative radial Ricci curvature, the function $r \to \frac{\mathrm{vol}(B(p,r))}{w_n r^n}$ is monotone non-increasing, which has been proved by Shiohama in [15].

3 Proof of the main theorem

In fact, we could prove the following more general theorem than Theorem 1.3.

Theorem 3.1. *Let (M,g) be an n-dimensional complete noncompact Riemannian manifold with $Ric_M^{(k)} \geq 0$ $(1 \leq k \leq n-1)$. Suppose that $K_p^{min} \geq -C$ for some point $p \in M$ and some positive constant C. If*

$$\limsup_{r\to\infty}\left[\frac{\mathrm{vol}(B(p,r))}{r^{1+\frac{1}{k+1}}\cdot v_p(r)}\right] \leq c_5\left(\frac{\log 2}{8\sqrt{C}}\right)^{\frac{k}{k+1}}, \tag{3.1}$$

where $c_5 = 2^{-5}c_2^{-1}$ is a positive constant depending only on c_2, and $v_p(r) := v_p(2,r)$, then M has finite topological type.

Proof. We use a similar method as that of theorem 2.2 in [1] to prove our theorem. By the isotopy Lemma 2.1, we know that if we want to prove the complete Riemannian manifold M is of finite topological type, we only need to show that there are no critical points outside a compact subset with respect to a fixed point $p \in M$. Take an arbitrary point $x(\neq p) \in M$ and set $r = d(p,x)$, which implies $x \in S(p,r)$. Since $c_5 = 2^{-2}c_4^{-1} = 2^{-5}c_2^{-1}$, $\limsup\limits_{r\to\infty}\left(\frac{r}{r+2}\right)^{n-1} = 1$,

$$\limsup_{r\to\infty}(r+2)\log\left(\frac{r+2}{r-2}\right) = \lim_{r\to\infty}\left(\frac{r+2}{r-2}\right)\cdot\lim_{r\to\infty}\log\left(1+\frac{4}{r-2}\right)^{(r-2)} = 4,$$

then our assumption (3.1) enables us to find a small number $\varepsilon > 0$ and a sufficiently large $r_1 > 4$ such that for any $r \geq r_1$, we have

$$\frac{\mathrm{vol}(B(p,r))}{r^{1+\frac{1}{k+1}}\cdot v_p(r)} < c_4^{-1}\left[\left(\frac{r}{r+2}\right)^{n-1}-\varepsilon\right]\left[\frac{1}{(r+2)\log(\frac{r+2}{r-2})}-\varepsilon\right]\left(\frac{\log 2}{8\sqrt{C}}-\varepsilon\right)^{\frac{k}{k+1}}. \tag{3.2}$$

On the other hand, since

$$\lim_{r\to\infty}\log\left(\frac{2}{1+e^{-2\sqrt{C}r}}\right) = \log 2,$$

there is a sufficiently large r_2 such that

$$\frac{\log\left(\frac{2}{1+e^{-2\sqrt{C}r}}\right)}{8\sqrt{C}} > \frac{\log 2}{8\sqrt{C}}-\varepsilon, \qquad \forall r \geq r_2. \tag{3.3}$$

Let $r_0 = \max\{r_1, r_2\}$, then from (3.2) and (3.3) we have

$$\frac{\mathrm{vol}(B(p,r))}{r^n\cdot v_p(r)} < c_4^{-1}\frac{r^{\frac{1}{k+1}}}{(r+2)^n}\left[\log\left(\frac{r+2}{r-2}\right)\right]^{-1}\cdot\left[\frac{1}{8\sqrt{C}}\log\left(\frac{2}{1+e^{-2\sqrt{C}r}}\right)\right]^{\frac{k}{k+1}}, \tag{3.4}$$

for any $r \geq r_0$. By Lemma 2.4, we could obtain

$$d(x,R_p) < r^{\frac{1}{k+1}}\cdot\left[\frac{1}{8\sqrt{C}}\log\left(\frac{2}{1+e^{-2\sqrt{C}r}}\right)\right]^{\frac{k}{k+1}}, \tag{3.5}$$

for any $r \geq r_0$. So, we can find a ray $\gamma : [0,\infty) \to M$ emanating from p and satisfying

$$s := d(x,\gamma) < r^{\frac{1}{k+1}} \cdot \left[\frac{1}{8\sqrt{C}} \log \left(\frac{2}{1+e^{-2\sqrt{C}r}} \right) \right]^{\frac{k}{k+1}}, \tag{3.6}$$

for any $r \geq r_0$. We can find a point $q \in \gamma$ such that $d(x,q) = d(x,\gamma)$, moreover, by (3.6), $d(x,q) < r$. Additionally, by triangle inequality, we know

$$\min(d(p,x), d(\gamma(t),x)) = r, \qquad \forall t \geq 2r. \tag{3.7}$$

Therefore $q \in \gamma((0,2r))$ and $d(x,\gamma|_{[0,2r]}) = s$. Then by Lemma 2.2, (3.6), and the fact $e_{p,\gamma}(x) \leq e_{p,\gamma(t)}(x)$ for any $t > 0$, we can obtain

$$e_{p,\gamma}(x) \leq e_{p,\gamma(2r)}(x) \leq 8 \left(\frac{s^{k+1}}{r} \right)^{\frac{1}{k}} < \frac{1}{\sqrt{C}} \log \left(\frac{2}{1+e^{-2\sqrt{C}r}} \right). \tag{3.8}$$

So, by Lemma 2.3 and (3.8), x is not a critical point of p. This implies p has no critical point out of a compact subset $\overline{B(p,r_0)}$. Hence, M has finite topological type. Our proof is finished. $\square$

Corollary 3.2. *Theorem 1.3 is true.*

References

[1] C. Xia, Open manifolds with nonnegative Ricci curvature and large volume growth, *Comment. Math. Helv.* **74** (1999) 456–466.

[2] C. Xia, Large volume growth and the topology of open manifolds, *Math. Z.* **239** (2002) 515–526.

[3] H. Zhan, The manifolds with nonnegative Ricci curvature and collapsing volume, *Proceedings of the American Mathematical Society* **135** (2007) 1923–1927.

[4] Z. Shen and G.Wei, Volume growth and finite topological type, *Proc. Sympos. in Pure Math.* **54** (1993) 539–549.

[5] J. Cheeger and D. Gromoll, On the structure of complete manifolds of nonnegative Ricci curvature, *Ann. of Math.* **96** (1972) 413–443.

[6] R. L. Bishop, R. J. Crittenden, Geometry of manifolds, *Academic Press*, New York(1964).

[7] M. Gromov, J. Lafontaine, P. Pansu, Structures métrique pour les variétes Riemanniennes. Cédic/Fernand, Nathan, Paris(1981).

[8] P. Li, Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature, *Ann. of Math.* **124** (1986) 1–21.

[9] M. Anderson, On the topology of complete manifolds of nonegative Ricci curvature, *Topology* **29** (1990) 41–55.

[10] P. Petersen, Comparison geometry problem list, Riemannian geometry (Waterloo, ON, 1993), 87–115, Fields Inst. Monogr., 4, Amer. Math. Soc., Providence, RI(1996).

[11] M. do Carmo, C. Y. Xia, Ricci curvature and the topology of open manifolds, *Math. Ann.* **316** (2000) 391–400.

[12] K. Grove and K. Shiohama, A generalized sphere theorem, *Ann. of Math.* **106** (1977) 201-211.

[13] U. Abresch, D. Gromoll, On complete manifolds with nonnegative Ricci curvature, *J. Amer. Math. Soc.* **3** (1990) 355–374.

[14] Z. Shen, On complete manifolds of nonnegative kth-Ricci curvature, *Trans. Amer. Math. Soc.* **338** (1993) 289–310.

[15] K. Shiohama, Comparison theorems for manifolds with radial curvature bounded below. *Josai. Math. Mono.* **3** (2001) 81–91.